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ORBITALLY QUANTIZED VORTEX STATES OF LASER RADIATION AND PHOTONIC SUPERFLUIDITY

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The existence of the phenomenon of photonic superfluidity in certain laser systems is demonstrated as is the identity of orbitally quantized excitations of the photonic Bose condensate to vortex lines in superfluid He and Abrikosov's vortex lines in type-II superconductors.

The problem of propagation of spiral laser radiation in various nonlinear media has been studied in detail in recent years [1-7]. The main difference of spiral light beams from conventional laser radiation is that they have a tubular intensity distribution, and the electric field rotates about the axis of propagation of the radiation. As I have shown earlier, this is associated with a new characteristic of spiral radiation, the topological charge $m = 0, \pm 1, \pm 2, \ldots$. The case of $m = 0$ corresponds to a conventional laser beam. The angular rotation velocity Ω of the electromagnetic field is determined by the frequency of the light ω and the topological charge m: $\Omega = \omega/m$, where $m \neq 0$.

Another important feature of spiral beams propagating in the self-waveguide regime is the fact that diffraction divergence is completely absent. This unusual, at first glance, property is connected with the optical nonlinearity of the medium. For certain types of nonlinear media, when the power integral of the spiral beam exceeds the threshold value, nonlinear interaction with the medium can lead to compression of the light beam and total compensation of diffraction phenomena. Spiral optical beams are in many ways similar to such macroscopic coherent phenomena as superfluidity and superconductivity, which has been stated several times in our previous works [4-6]. Investigation of the problem shows that spiral laser radiation propagating in a nonlinear medium is a rotationally quantized excitation of the superfluid photonic Bose condensate.

On the other hand, it is known that vortex lines excited in superfluid He II and Abrikosov's vortex threads in type II superconductors are orbitally quantized excitations of the Bose condensate. Thus, we come to the conclusion that the three above-mentioned phenomena are physically identical. It should be noted, however, that identity of their physical mechanism does not exclude certain distinctive features inherent in each of them. In particular, the Bose particles in all three cases are different. In the case of superfluidity, the Bose condensate consists of Cooper pairs.

Another important distinctive feature of the photonic Bose condensate is that the particles have zero mass, and, therefore, they all move with a velocity close to the speed of light in vacuum. In what follows, we will show that the topological charge m of a spiral beam [1] is actually a quantum number that characterizes the rotational state of individual photons of the spiral radiation field.

Quantized Field of Laser Radiation and the Theory of Superfluidity. We present a theory of propagation of spiral laser radiation in a nonlinear medium in quantum-hydrodynamic form. Let us write the strength of the electric field E in the following form:

$$\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- , \quad \mathbf{E}_+ = \frac{1}{2} \sum_{\lambda=0,\pm 1} \mathbf{e}_\lambda F_\lambda \exp(-i\omega t) , \tag{1}$$

$$\mathbf{E}_- = \mathbf{E}_+^* , \quad \mathbf{e}_\lambda = \frac{1}{\sqrt{2}} (\mathbf{n}_1 + i\lambda \mathbf{n}_2) , \quad \mathbf{e}_0 = \mathbf{n}_3 , \tag{2}$$

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where $\mathbf{e}_{\pm}$ are circular polarization vectors. When free charges are absent the wave equation for the transverse electric field in a nonlinear medium takes the form

$$\nabla^2 \mathbf{E}_+ - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\varepsilon(|F|^2)\,\mathbf{E}_+) = 0\,, \tag{3}$$

where

$$\varepsilon(|F|^2) = \sum_{s=0}^{\infty} \varepsilon_{2s}\,|F|^{2s}\,, \quad |F|^2 = \sum_{\lambda} |F_\lambda|^2\,, \tag{4}$$

and it should be noted that the condition $\nabla \mathbf{E}_+ = 0$ is approximately satisfied. Let us introduce the following functions Ψ_λ:

$$\Psi_\lambda = \frac{1}{\sqrt{a}} F_\lambda\,, \quad a = \frac{8\pi\hbar\omega}{\varepsilon_0}\,, \quad \lambda = \pm 1\,, \tag{5}$$

which, as will be shown in what follows, can be treated as wave functions of the collective photonic state. Indeed, it follows from (5) that the energy density W of the electromagnetic field of the radiation has the form

$$W = \hbar\omega \sum_{\lambda=\pm 1} N_\lambda\,, \quad N_\lambda = |\Psi_\lambda|^2\,. \tag{6}$$

Based on Eq. (3), in an approximation of slowly varying amplitudes F_λ, we arrive at the Schrödinger-type equation

$$i\hbar\frac{\partial}{\partial t}\Psi_\lambda = H(N)\,\Psi_\lambda\,, \quad H(N) = -\frac{\hbar^2}{2M_0}\nabla^2 + U(N)\,, \tag{7}$$

$$U(N) = -\frac{\hbar\omega}{2\varepsilon_0}\varepsilon(aN)\,, \quad N = \sum_{\lambda=\pm 1} N_\lambda\,, \tag{8}$$

where the effective mass M_0 is introduced using the relationship $\hbar\omega = M_0 c_0^2$, $c_0 = c/\sqrt{\varepsilon_0}$. In accordance with quantum mechanics, we determine the flux density $\mathbf{J}_\lambda$ and the velocity $\mathbf{v}_\lambda$:

$$\mathbf{J}_\lambda = \frac{i\hbar}{2M_0}(\Psi_\lambda \nabla \Psi_\lambda^* - \Psi_\lambda^* \nabla \Psi_\lambda)\,, \quad \mathbf{v}_\lambda = N^{-1}\mathbf{J}_\lambda = \frac{\hbar}{M_0}\nabla\Theta_\lambda\,, \tag{9}$$

where $\Psi_\lambda = \sqrt{N_\lambda}\exp(i\Theta_\lambda)$. Using Eq. (9) we can represent the wave Eq. (7) in hydrodynamic form:

$$\frac{\partial}{\partial t}N_\lambda + \nabla(N_\lambda \mathbf{v}_\lambda) = 0\,, \tag{10}$$

$$\frac{\partial}{\partial t}\mathbf{v}_\lambda + \nabla\left(\mu_\lambda + \frac{1}{2}v_\lambda^2\right) = 0\,, \tag{11}$$

$$\mu_\lambda = \frac{1}{M_0}(U_0(N_\lambda) + U(N))\,, \quad U_0(N_\lambda) = -\frac{\hbar^2}{2M_0\sqrt{N_\lambda}}\nabla^2\sqrt{N_\lambda}\,. \tag{12}$$

It should be noted that Eq. (11) can also be written in a form that coincides with the hydrodynamic equation of motion of an ideal fluid

$$\frac{\partial}{\partial t}\mathbf{v}_\lambda + (\mathbf{v}_\lambda\nabla)\,\mathbf{v}_\lambda = -\frac{1}{M_0}\nabla(U_0(N_\lambda) + U(N))\,. \tag{13}$$

Integration of Eq. (13) leads to an equation for the velocity potential Θ_λ:

$$\hbar \frac{\partial}{\partial t} \Theta_\lambda + \frac{1}{2} M_0 v_\lambda^2 + U_0(N_\lambda) + U(N) = 0 . \tag{14}$$

It is interesting to note that Eqs. (10)-(12) coincide with the Landau equations for the superfluid component of He II in the theory of superfluidity [8]. Equations (10)-(14) also coincide with the hydrodynamic form of the equations of the theory of type-II superconductors [9, 10]. In addition, the theorem of conservation of velocity circulation follows from (13):

$$D_\lambda \Gamma_\lambda = 0 , \quad D_\lambda = \frac{\partial}{\partial t} + \mathrm{v}_\lambda \nabla_\lambda , \quad \Gamma_\lambda = \oint_{c_\lambda} \mathrm{v}_\lambda \, d\mathrm{l} , \tag{15}$$

where c_λ is an arbitrary closed 1-connected fluid contour. Calculation of the circulation Γ_λ for a spiral laser beam [1-7] leads to the following result:

$$\Gamma_\lambda = \oint_{c_\lambda} \mathrm{v}_\lambda \, d\mathrm{l} = \frac{2\pi\hbar}{M_0} m_\lambda , \quad m_\lambda = 0, \pm 1, \pm 2, \ldots \tag{16}$$

where we used the definition of velocity (9). Here c_λ is a positively oriented 1-connected closed contour containing the z axis. Relation (16) coincides exactly with the quantization rule for velocity circulation in the theory of superfluidity [8, 9]. It should be noted that quantization condition (16) also holds in the theory of type-II superconductors, and v_λ in this case has the meaning of the canonical velocity of Cooper pairs [8, 9]. It can be easily seen that relation (16) is also equivalent to the Bohr-Sommerfeld quantization equation:

$$\oint_{c_\lambda} \mathrm{p}_\lambda d\mathrm{q}_\lambda = 2\pi \hbar m_\lambda , \quad m_\lambda = 0, \pm 1, \pm 2, \ldots , \tag{17}$$

where $\mathrm{p}_\lambda = M_0 \mathrm{v}_\lambda$ and $\mathrm{q}_\lambda (d\mathrm{q}_\lambda = d\mathrm{l}, \lambda = \pm 1)$ are canonical momenta and coordinates. It follows particularly from Eq. (17) that the quantity $\hbar m_\lambda$ is the projection of the orbital moment of photons of the spiral laser beam onto the quantization axis z. Thus, the topological charge m_λ of the spiral laser beam coincides with the orbital quantum number of photons of the vortex electromagnetic field. According to (7), (8), and (16) the tangential projection of the velocity v_λ and the angular velocity Ω_λ of the spiral laser beam are as follows:

$$(\mathrm{v}_\lambda)_\varphi = \frac{\hbar m_\lambda}{M_0 r} , \quad \Omega_\lambda = \frac{\omega}{m_\lambda} , \tag{18}$$

where $m_\lambda \neq 0$. It should be noted that the formula for $(\mathrm{v}_\lambda)_\varphi$ coincides exactly with the corresponding results of the theory of superfluidity and superconductivity. Taking into account Eq. (18) and results of [6], we calculate the rotational energy E_R of the spiral beam

$$\frac{d}{dz} E_R = \frac{c^2}{8\pi\omega^2} \sum_{\lambda = \pm 1} |m_\lambda| \kappa_\lambda(z) I_\lambda , \quad I_\lambda = \int\int |F_\lambda|^2 \, dx dy . \tag{19}$$

Similarly, we find an expression for the projection of the angular moment L_z onto the z axis of the spiral laser beam:

$$\frac{d}{dz} L_z = \frac{\varepsilon}{8\pi\omega} \sum_{\lambda = \pm 1} m_\lambda I_\lambda . \tag{20}$$

We consider as an example the case when the intensity distribution of the laser beam does not depend on z. As has been shown in [2-6], under certain conditions this regime of propagation of spiral laser radiation is actually realized. Let the spiral beam have the spirality $\lambda = +1$. Then the solution of Eq. (7) has the form

$$\Psi_{+1}(r, \varphi, t) = f_m(r) \exp\left[i(m\varphi + kz - \omega t + \Phi_0)\right] . \tag{21}$$

Substituting (21) into (7) and taking into account that $\Psi_{-1} = 0$ we arrive at an equation for the real-valued amplitude $f_m(r)$ [1]:

$$r^{-1} \frac{d}{dr} \left(r \frac{d}{dr} f_m \right) - \frac{m^2}{r^2} f_m + \frac{\omega^2}{c^2} (\varepsilon (a f_m^2) - \varepsilon_0) f_m = 0 , \tag{22}$$

where

$$\varepsilon (a f_m^2) = \varepsilon_0 + \varepsilon_2 a f_m^2 + \varepsilon_4 a^2 f_m^4 + \dots , \quad m = \pm 1, \pm 2, \dots . \tag{23}$$

If we retain only the first two terms in expansion (23), Eq. (23) will coincide with the Ginzburg-Pitaevskii equation for quantized vortex lines in the theory of superfluidity of a nearly ideal Bose condensate at zero temperature [8]. It should be noted that Eq. (22) also describes Abrikosov's quantized vortex threads in the vicinity of the singularity line [10].

Orbital Spiral States of the Rotationally Excited Photonic Bose Condensate. The above results demonstrate that the theory of propagation of spiral laser beams in a nonlinear medium is formally identical to the theory of superfluidity of He II and the theory of type-II superconductors. This suggests that a spiral laser beam is an orbitally quantized vortex excitation of the Bose condensate. In order to prove this statement, we should first define the photonic states of the radiation field corresponding to orbital excitation of vacuum [11, 12]. Single-photon states of the radiation field are described by wave Eq. (3) at $\varepsilon = \varepsilon_0 = \text{const}$ and have the form

$$\mathbf{E}_+^N = \frac{1}{2} \mathbf{F}_N \exp (- i\omega t) , \quad \nabla \mathbf{E}_+^N = 0 , \tag{24}$$

where N is the complete set of photon quantum numbers. From (3) and (24) we have an equation for the amplitude $\mathbf{F}_N$:

$$\nabla^2 \mathbf{F}_N + k^2 \mathbf{F}_N = 0 , \quad \nabla \mathbf{F}_N = 0 , \tag{25}$$

where $k = \sqrt{\varepsilon_0}\omega / c$. It should be noted that the set of photon quantum states is determined by a system of linear eigenvalue equations:

$$S_z \mathbf{F}_{NT} = \lambda \mathbf{F}_{NT} , \quad P_z \mathbf{F}_N = \hbar\kappa \mathbf{F}_N , \quad L_z \mathbf{F}_{NT} = \hbar m \mathbf{F}_{NT} . \tag{26}$$

Here S_z is the photon polarization operator, P_z and L_z are the operators of theprojections of the photon momentum and orbital moment onto the z axis:

$$S_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad P_z = - i\hbar \frac{\partial}{\partial z} ,$$
$$L_z = - i\hbar \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) = - i\hbar \frac{\partial}{\partial \varphi} . \tag{27}$$

Here $\mathbf{F}_{NT}$ is the transverse component of the vector $\mathbf{F}_N$. $\mathbf{F}_N = \mathbf{F}_{NT} + \mathbf{F}_{NL}$, $\mathbf{F}_{NT}\,\mathbf{n} = 0$, where $\mathbf{n}$ is a unit vector directed along the z axis. It follows from (26) that the complete set of quantum numbers of the introduced photonic states is $N = (\lambda, \omega, \kappa, m)$, where λ is the photon spirality, $\hbar\omega$ is the photon energy, $\hbar\kappa$ is the projection of the momentum onto the z axis, and $\hbar m$ is the projection of the orbital moment onto the z axis. It can be shown [11, 12] that the solution of Eqs. (25) and (26) has the form:

$$\mathbf{F}_N (\mathbf{r}) = A_N \int_0^{2\pi} \frac{\mathbf{e}_\lambda (\vartheta, \varphi')}{\sqrt{(2\pi)^3}} \exp [i (k\mathbf{n} (\vartheta, \varphi') \mathbf{r} + (m + \lambda)\varphi')] \, d\varphi' . \tag{28}$$

Here we used the following parametrization of the circular-polarization vectors $\mathbf{e}_\lambda(\vartheta, \varphi)$:

$$\mathbf{e}_\lambda(\vartheta,\varphi) = \frac{\exp(-i\lambda\varphi)}{\sqrt{1+\cos^2\vartheta}} \begin{pmatrix} \cos\vartheta \\ i\lambda\cos\vartheta \\ -\sin\vartheta e^{i\lambda\varphi} \end{pmatrix}, \tag{29}$$

and it should be noted that

$$\mathbf{n}(\vartheta,\varphi) = \sin\vartheta\cos\varphi\mathbf{n}_1 + \sin\vartheta + \sin\varphi\mathbf{n}_2 + \cos\vartheta\mathbf{n}_3 .$$

Integration in (28) leads to an expression for the positive-definite part of the electric field of orbital spiral photonic states [11, 12]:

$$\mathbf{E}_+^N(r,\varphi,z,t) = C_N e^{i(m\varphi+\kappa z-\omega t)} \begin{pmatrix} \kappa k^{-1} J_m(qr) \\ i\lambda\kappa k^{-1} J_m(qr) \\ -i\lambda q k^{-1} e^{i\lambda\varphi} J_{m+\lambda}(qr) \end{pmatrix}. \tag{30}$$

Here C_N = const, κ and q are the longitudinal and transverse wave numbers:

$$\kappa = k\cos\vartheta, \quad q = k\sin\vartheta, \quad k^2 = \kappa^2 + q^2, \quad k = \frac{\omega}{c_0},$$

$J_m(qr)$ are Besssel functions of the first kind, $0 \le \vartheta \le \pi$ is the angle of the spherical system of coordinates, $m = 0, \pm 1, \pm 2, \ldots$ is the orbital quantum number of the photon.

The wave function of the photon (30) also satisfies the system of eigenvalue equations:

$$i\hbar\frac{\partial}{\partial t}\mathbf{E}_+^N = \hbar\omega\mathbf{E}_+^N, \quad P_z\mathbf{E}_+^N = \hbar\kappa\mathbf{E}_+^N,$$
$$M_z\mathbf{E}_+^N = \hbar(\lambda+m)\mathbf{E}_+^N, \quad K_\lambda\mathbf{E}_+^N = \lambda\mathbf{E}_+^N, \tag{31}$$

where

$$M_z = L_z + \hbar S_z, \quad K_{\pm 1} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}.$$

The vortex state of the laser radiation in a nonlinear medium can be represented in the form of a wave package of photonic states (30) with fixed spirality λ and orbital quantum number m. Taking into account (30), and the fact that $\sin\vartheta \ll 1$, we can neglect the longitudinal component of the field, which leads to the following expression:

$$\mathbf{E}_+ = \frac{\mathbf{e}_\lambda}{2} e^{i(m\varphi-\omega t)} \int_0^\infty \int_0^1 G_{\lambda m}(p,\mu,t) J_m(rp\sqrt{1-\mu^2}) \exp(i\mu pz)\, dp d\mu . \tag{32}$$

Here $\mu = \cos\vartheta$, and $G_{\lambda m}(p,\mu,t)$ is a function determined by the mean value of the annihilation operator $\hat{a}_{\lambda p\mu m}$ of an orbital spiral photon of the spiral beam. Expanding $\exp(i\mu pz)$ into a Bessel series, we can write wave package (32) for a complex amplitude F_λ as follows:

$$F_\lambda(r,\varphi,z,t) = e^{im\varphi} \sum_{n=-\infty}^{+\infty} \int_0^\infty F_{\lambda m}^n(r,p,t) J_n(pz)\, dp, \tag{33}$$

$$F_{\lambda m}^n(r,p,t) = i^n \int_0^{\pi/2} G_{\lambda m}(p,\cos\vartheta,t) J_m(rp\sin\vartheta) \exp(-in\vartheta) \sin\vartheta d\vartheta . \tag{34}$$

Inasmuch as photons of the spiral beam have vanishing energy dispersion, the complex-valued amplitude F_λ can be written as follows:

$$F_\lambda (r, \varphi, z, t) = e^{im\varphi} \int_0^1 g_{\lambda m} (\mu, t) J_m (rk \sqrt{1 - \mu^2}) \exp (i\mu kz) \, d\mu \,. \tag{35}$$

Expansion (33) in this case is transformed to a Neumann series:

$$F_\lambda (r, \varphi, z, t) = \frac{1}{2} C_{\lambda m}^0 (r, \varphi, t) J_0 (kz) + \sum_{n=1}^{\infty} C_{\lambda m}^n (r, \varphi, t) J_n (kz) \,, \tag{36}$$

where

$$C_{\lambda m}^n (r, \varphi, t) = \exp (im\varphi) (f_{\lambda m}^n (r, t) + (-1)^n f_{\lambda m}^{-n} (r, t)) \,, \tag{37}$$

$$(f_{\lambda m}^n (r, t) = i^n \int_0^{\pi/2} g_{\lambda m} (\cos \vartheta, t) J_m (rk \sin \vartheta) \exp (-in\vartheta) \sin \vartheta d\vartheta \,. \tag{38}$$

It is well known that only periodic functions can be expanded into Neumann series, and, therefore, the function F_λ is periodic with respect to z. This fact is in complete agreement with the theory of self-waveguide propagation of spiral laser beams [2-6]. In addition, here we found that the orbital quantum number m coincides with the topological charge of the spiral beam introduced in [1, 2].

Orbitally Quantized States of the Photonic Condensate in an Active Laser Medium. We consider rotational vortex states of the electromagnetic field in an active laser medium on the basis of the Maxwell–Bloch system of laser equations. In an approximation of slowly varying amplitudes the system can be written as follows:

$$\frac{\partial}{\partial t} E = i\Delta E - \alpha (1 + i\delta_0) E + F \,, \quad \frac{\partial}{\partial t} F = -\gamma (1 - i\delta_0) F + \gamma Q E \,,$$
$$\frac{\partial}{\partial t} Q = -\beta (Q - Q_0) - \frac{\beta}{2} (EF^* + FE^*) \,. \tag{39}$$

Here, for convenience, all the variables and constants are represented in dimensionless form. Complex-valued functions E and F describe the electric field and polarization of the active laser medium, Q is the population inversion, γ and β are the transverse and logitudinal relaxation times of the medium, δ_0 is the detuning from the atomic-transition frequency, and Δ is the Laplace operator in polar coordinates:

$$\Delta E = r^{-1} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} E \right) + r^{-2} \frac{\partial^2}{\partial \varphi^2} E \,.$$

Pumping and losses of radiation in the laser are determined by the parameters Q_0 and α. In what follows, we consider a definite type of laser with boundary and initial conditions of the following form:

$$\left. \frac{\partial E}{\partial r} \right|_{r=R} = 0 \,, \quad E|_{t=0} = E_0 (r, \varphi) \,, \quad F|_{t=0} = F_0 (r, \varphi) \,, \quad Q|_{t=0} = Q_0 \,. \tag{40}$$

Here $E_0(r, \varphi)$ and $F_0(r, \varphi)$ are determined by fluctuation processes in the laser, and R is the boundary value of the dimensionless variable r.

The simplest nontrivial solution of system (39) and (40) at $Q_0 = \text{const}$ is expressed by a stationary uniform distribution:

$$E = \left(\frac{Q_0}{\alpha} - 1 - \delta_0^2 \right)^{1/2} \exp (i\Phi_0) \,, \quad \frac{Q_0}{\alpha} > 1 + \delta_0^2 \,, \tag{41}$$

where $\varphi_0 = \text{const}$. Single-vortex solutions at arbitrary values of the topological charge $m = \pm 1, \pm 2, \pm 3, \ldots$ are as follows:

$$E = E_m \exp(-i\Omega_m t), \quad F = F_m \exp(-i\Omega_m t), \quad \frac{\partial}{\partial t} Q = 0. \tag{42}$$

Substituting (42) into (39), we arrive at a nonlinear equation for eigenfunctions E_m and eigenfrequencies Ω_m:

$$\Omega_m E_m = -\Delta E_m - \alpha (i - \delta_0) E_m + \frac{(1 - \delta_m) Q_0 E_m}{1 + \delta_m^2 + |E_m|^2}, \tag{43}$$

where

$$\delta_m = \delta_0 + \frac{\Omega_m}{\gamma}, \quad E_m(r, \varphi) = f_m(r) \exp[i(m\varphi + \Phi_m(r))]. \tag{44}$$

In accordance with (40), the boundary conditions for $f_m(r)$ and $\Phi_m(r)$ are as follows:

$$\left(\frac{d}{dr} f_m(r)\right)_{r=R} = 0, \quad \left(\frac{d}{dr} \Phi_m(r)\right)_{r=R} = 0. \tag{45}$$

Introduction of orbital spiral photonic states (30) makes it possible to represent the amplitude $E_m(r, \varphi)$ in the form of a wave package:

$$E_m(r, \varphi) = \exp(im\varphi) \int_0^a g_m(\mu) J_m(\mu r) \, d\mu. \tag{46}$$

Let us consider the case when the wave package (46) has low dispersion with respect to the transverse wave numbers $\mu \sim k \sin \vartheta$; then the amplitude $f_m(r)$ can be presented as follows:

$$f_m(r) = C_m J_m\left(\mu_m \frac{r}{R}\right), \quad m = \pm 1, \pm 2, \pm 3, \ldots, \tag{47}$$

where μ_m is the first positive root of the equation $dJ_m(r)/dr = 0$. The constant C_m is determined by substituting (47) into (43) and passing to the limit $r \to R$. Thus, using (42), (44), (45), and (47) we find the approximate solution

$$E(r, \varphi, t) = C_m J_m\left(\mu_m \frac{r}{R}\right) \exp[i(m\varphi - \Omega_m t + \Phi_m(r))], \tag{48}$$

$$C_m = J_m^{-1}(\mu_m) \left(\frac{Q_0}{\alpha + \Phi_m''(R)} - 1 - \delta_m^2\right)^{1/2}, \quad \delta_m = \delta_0 + \frac{\Omega_m}{\gamma}. \tag{49}$$

The phase $\Phi_m(r)$ can be depermined from Eq. (43) with allowance for (45), (47), and (49). To obtain an equation for eigenfrequencies Ω_m, one can simply substitute (17) into (43) and pass to the limit $r \to 0$:

$$\frac{Q_0 \delta_m}{1 + \delta_m^2} + \gamma \delta_m = (\alpha + \gamma) \delta_0 + \frac{\mu_m^2}{R^2}, \quad \Omega_m = \gamma(\delta_m - \delta_0). \tag{50}$$

Comparison of the approximate analytical solution (48) at $\Phi_m(r) = \text{const}$ with the numerical solution of the system (39) and (40) in the stationary regime shows that in the vicinity of resonance $\delta_m = 0$ at $m = \pm 1, \pm 2, \pm 3$ the solutions coincide with each other up to the accuracy of the numerical experiment [12]. Thus, in numerical calculations [12] the analytical (48) and (49) and numerical solutions coincided to the third decimal place, and the discrepancy was less than 0.1%. This means that in the vicinity of points $\delta_m = 0$ the vortex state of the laser radiation, according to (48) and (49), is actually a condenstate of orbital spiral photons (30) in the active laser medium. It should be noted that vortex states of the radiation field were considered in [13-15].

Formulas (41) and (48) can also be expressed in terms of ordinary dimensional variables:

$$\mathbf{E}_+ = \mathbf{e}_\lambda E_0 \left(\frac{Q_0}{\alpha} - 1 - \delta_0^2\right)^{1/2} \exp\,[i\,(k_0 z - \omega_0 t + \Phi_0)]\,, \tag{51}$$

$$\mathbf{E}_+ = \mathbf{e}_\lambda E_0 C_m J_m \left(\mu_m \frac{r}{r_0}\right) \exp\,[i\,(m\varphi + \kappa z - \omega t + \Phi_0)]\,, \tag{52}$$

where r, z, and t are dimensional quantities, E_0 is a parameter of the laser system with the dimensionality of the electric-field strength, r_0 is the radius of the active medium of the laser, k_0 and κ are wave numbers, Φ_0 = const, and $\delta_m = 0$. Formulas (51) and (52) obviously express only the waves propagating in the positive direction of the z axis of the active laser medium. In the case of a conventional laser, similar waves also propagate in the opposite direction.

Solution (51) holds when $R < R_1$, where R_1 is the bifurcation point determined from (50) at $\Omega_m = 0$. When $R > R_1$, a transition from distribution (51) to (52) takes place, and it should be noted that $m = \pm 1$. With passage through the bifurcation point R_1, a frequency shift $\Delta\omega_R$ can be observed:

$$\omega = \omega_0 + \Delta\omega_R\,, \quad \Delta\omega_R = \frac{\Omega_m}{t_0}\,. \tag{53}$$

here t_0 is the characteristic time parameter of the laser, which is used when the time in the system of equations (39) is made dimensionless: $t/t_0 \to t$. It follows from (50) and (53) that at the points of vortex resonance $\delta_m = 0$, the frequency shift $\Delta\omega_R$ is described by the expression:

$$\Delta\omega_R = \frac{\gamma\,|\delta_0|}{t_0}\,. \tag{54}$$

It should be noted that the ground state of the field in the laser (51) is of fundamental significance from the viewpoint of the theory being developed here. Indeed, this is a superfluid photonic state of the Bose condensate that is created in an active laser medium when $R < R_1$ and $Q_0/\alpha > 1 + \delta_0^2$. This is a ground superfluid state of a multiphoton quantum system with all quanta having equal values of the wave vector **k** and spirality λ. These sates, as is known, are described by plane waves with rather definite normalization of the resulting wave (51).

On the other hand, the vortex quantum state of the radiation field (52) is completely identical, from the physical point of view, to orbitally quantized vortex lines in the theories of superfluidity and type-II superconductivity [8-10]. Note that the orbitally quantized radiation (52) corresponds to the wave package (32) with zero dispersion of the logitudinal and transverse wave numbers $\kappa = k\cos\vartheta$, $q = k\sin\vartheta$ of orbital spiral photonic states (30). Indeed, it follows from a comparison of expressions (32) and (52) that

$$G_{\lambda m}(p, \mu, t) = \alpha_{\lambda m}\delta\,(p - k)\,\delta\,((r_0 k)^{-2}\mu_m^2 - 1 + \mu)\,. \tag{55}$$

This means that all orbital spiral photons of the vortex state (52) are in the same quntum state $N = (\lambda, \omega, \kappa, m)$.

Therefore, we can state that the superfluid Bose condensate (51) can be created under certain conditions in laser systems considered in the present article. Orbitally quantized excitations of the superfluid photonic condensate (51) at resonance points $\delta_m = 0$ are desribed with high accuracy by expression (51) and, as has been found, are identical to vortex lines excited in He II and type-II superconductors. Thus, we come to the existence of type-III superfluidity, namely, photonic superfluidity. In this case the self-waveguide propagation of the spiral radiation in nonlinear media and the vortex states of the electromagnetic field in the active laser medium represent orbitally quantized excitations of the superfluid Bose condensate with a definite orbital quantum number $m = \pm 1, \pm 2, \ldots$.

Index Theorem. It has been demonstrated in numerical experiments [12] that, along with vortex solutions with various topological charges, N-vortex solutions of system (39) and (40) exist. The N-vortex stationary solutions of Eqs. (39) and (40) can be presented in the following form [16]:

$$E(r, \varphi, t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_{nm}(t) z^n \overset{*}{z}^m , \tag{56}$$

where $z = r \exp(i\varphi)$, $\overset{*}{z} = r \exp(-i\varphi)$. Since the field strength vanishes at the center of each of the vortices, the N-vortex solution has the form [16]:

$$E(r, \varphi, t) = E_0 \exp(K(r, \varphi, t)) \prod_{s=1}^{N} (z - z_s)^{\nu_s} (\overset{*}{z} - \overset{*}{z}_s)^{\mu_s} , \tag{57}$$

where $K(r, \varphi, t)$ is a totally defined function that has an expansion over the variables z and $\overset{*}{z}$ similar to (56). In addition, ν_s = const and μ_s = const are nonnegative integers in the stationary regime, and $\nu_s + \mu_s > 0$.

It is important to note that function (56) and (57) is not analytical with respect to z, since the Cauchy-Rieman condition is not satisfied. Let us prove that the index of the function $E(r, \varphi, t)$ equals the sum of topological charges of the vortices enclosed by the contour C:

$$\text{Ind}\, E(r, \varphi, t)|_c \equiv \frac{1}{2\pi} \oint_c d\Phi = \sum_{s=1}^{n} m_s . \tag{58}$$

Here $\Phi = \arg E(r, \varphi, t)$, $m_s = \nu_s - \mu_s$, c is an arbitrary positively directed closed 1-connected contour belonging to the region $r < R$, z_s ($s = 1, 2, ..., n$; $n \le N$) are zeros of the function $E(r, \varphi, t)$ belonging to the region enclosed by contour c. In order to prove the theorem we will use a formula that follows from (57):

$$E^{-1} \nabla E = (\mathbf{e}_x = i\mathbf{e}_y) \sum_{s=1}^{N} \frac{\nu_s}{z - z_s} + (\mathbf{e}_x - i\mathbf{e}_y) \sum_{s=1}^{N} \frac{\mu_s}{\overset{*}{z} - \overset{*}{z}_s} + \nabla K . \tag{59}$$

Here $\mathbf{e}_x$ and $\mathbf{e}_y$ are unit basis vectors, $\nabla = \mathbf{e}_x \partial/\partial x + \mathbf{e}_y \partial/\partial y$. After integration of expression (59) along contour C we obtain:

$$\oint_c E^{-1} \nabla E ds = \sum_{s=1}^{N} \nu_s \oint_c \frac{dz}{z - z_s} + \sum_{s=1}^{N} \mu_s \oint_c \frac{d\overset{*}{z}}{\overset{*}{z} - \overset{*}{z}_s} = 2\pi i \sum_{s=1}^{n} (\nu_s - \mu_s) . \tag{60}$$

Here we took into account that

$$\oint_c \nabla K ds = \oint_c dK = 0 , \tag{61}$$

since $K(r, \varphi, t)$ is single-valued within the region $r < R$.

On the other hand, assuming that $E = |E| \exp(i\Phi)$, we find that

$$\frac{1}{2\pi i} \oint_c E^{-1} \nabla E ds = \frac{1}{2\pi i} \oint_c \nabla \ln |E|\, ds + \frac{1}{2\pi} \oint_c \nabla \Phi ds = \frac{1}{2\pi} \oint_c d\Phi , \tag{62}$$

since the function $\ln |E|$ is single-valued, and, consequently,

$$\oint_c \nabla \ln |E|\, ds = 0 . \tag{63}$$

With allowance for (62), expression (60) can be presented in the form:

$$\frac{1}{2\pi} \oint_c d\Phi = \sum_{s=1}^{n} (\nu_s - \mu_s) = \sum_{s=1}^{n} m_s . \tag{64}$$

Thus, the theorem is proved.

Let c_s be an arbitrarily oriented 1-connected closed contour enclosing only the s-th zero of the function $E(r, \varphi, t)$. In this case, according to (64), we find that

$$m_s = \frac{1}{2\pi} \oint_{c_s} d\Phi . \tag{65}$$

Substituting (65) into (64), we obtain

$$\oint_c d\Phi = \sum_{s=1}^{n} \oint_{c_s} d\Phi . \tag{66}$$

We note that expressions (58) and (66) completely agree with results of numerical experiments for stationary N-vortex regimes [12].

Finally, we present the proof of the conservation law for circulation (15):

$$D_\lambda \Gamma_\lambda = \oint_{c_\lambda} (D_\lambda \mathrm{v}_\lambda)\, dl + \oint_{c_\lambda} \mathrm{v}_\lambda (D_\lambda dl) = - \oint_{c_\lambda} \nabla \mu_\lambda dl + \oint_{c_\lambda} \mathrm{v}_\lambda d\mathrm{v}_\lambda = \oint_{c_\lambda} d \left(\frac{1}{2} v_\lambda^2 - \mu_\lambda \right) = 0 . \tag{67}$$

Here c_λ is an arbitrary 1-connected closed fluid contour. In deriving expression (67) we used Eq. (13) and the single-valuedness of the function $1/2 v_\lambda^2 - \mu_\lambda$ at an arbitrary point of space.

Conclusion. The existence of homogeneous solution (51) of system of Eqs. (39) and (40) is of fundamental importance for the theory presented and is also substantiated by numerical experiments [12]. This regime takes place only when $R < R_1$, where R_1 is the first bifurcation point. When $R > R_1$, transition from regime (51) to vortex regime (52) takes place, and it should be noted that $m = \pm 1$. The bifircation point R_1 is found from (50) if one sets $\Omega_m = 0$:

$$R_1 = \mu_1 \left(\frac{Q_0 \delta_0^2}{1 + \delta_0^2} - \alpha \delta_0 \right)^{-1/2} . \tag{68}$$

The results of numerical calculations [12, 16] agree with the analytical value of (68) for the first bifurcation with an error of less than 1%.

The value of the parameter R corresponding to vortex resonance can also be found from (50) at $\delta_m = 0$:

$$R = \mu_m \, [(\alpha + \gamma) \, |\delta_0| \,]^{-1/2} , \quad \Omega_m = \gamma \, |\delta_0| \tag{69}$$

and is reproduced with high accuracy in numerical experiments [12, 16]. In addition, it can be shown that the power of the rotating laser radiation field at the point of the vortex resonance (69) has a maximum and is determined by the formula:

$$P = \int_0^{2\pi} d\varphi \int_0^R |E|^2 \, r dr = \frac{\pi \, (\mu_m^2 - m^2)}{(\alpha + \gamma) \, |\delta_0|} \left(\frac{Q_0}{\alpha} - 1 \right) . \tag{70}$$

This power, in particular, is higher than the power in the homogeneous regime (51).

As has been shown, the homogeneous stationary state of the laser radiation field (51) comprises the photonic Bose condensate, which can be considered a superfluid quantum state of the radiation field with the lowest energy. With passage through bifurcation point R_1, the superfluid state of the multiparticle quantum system becomes unstable, and an orbitally quantized state of the radiation field is excited.

At the resonance point (69), dispersion of the wave package that describes the vortex field vanishes. Due to this fact, the orbitally quantized state of the radiation field is described with high accuracy by expression (52) in the point of vortex resonance $\delta_m = 0$. It follows from the theory presented that vortex state (52) is physically identical to quantized vortex lines in superfluid He II and Abrikosov's lines in type-II superconductors. Indeed, the resonance vortices described by expression (52) are multiphoton states of the radiation field with all quanta in orbital spiral state (30). Thus, vortex state (52) is the Bose condensate of orbital spiral photons (30).

Numerical solution of the system of Eqs. (39) and (40) has shown that for rather high values of R, when a considerable number of vortices is created, a stationary regime that comprises a vortex lattice with a square cell can be established in the active laser medium. In such a multivortex lattice, neighboring vortices have opposite topological charges $m = \pm 1$. This vortex photonic lattice is in many ways similar in its properties to the vortex lattice formed by Cooper pairs in type-II superconductors [10].

The fact that the lattice is square in the former case and triangular in the latter case is explained by the difference in interactions. Thus, the existence of a vortex lattice in an active laser medium can be considered as an additional substantiation of photonic superfluidity, which I have called type-III superfluidity [11, 12].

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NOTATION

E, strength of the electric field; N_λ, photon density; Θ_λ, velocity potential; v_λ, local velocity of the radiation field; μ_λ, chemical potential of photons; Γ_λ, circulation of the velocity; m_λ, topological charge; Ω_λ, angular velocity of the radiation field; E, complex-valued strength of the electric field; F, complex-valued polarization of the active laser medium; α, radiation losses in the laser; γ, transverse relaxation time; β, longitudinal relaxation time; δ_0, detuning from the frequency of the quantum transition; Q, inversion of laser levels; Q_0, pumping parameter of the laser energy; κ, transverse wave number of photons; q, longitudinal wave number of photons; ω, frequency of light; P, power of laser radiation; Φ, phase of the electromagnetic field.

REFERENCES

1. V. I. Kruglov and R. A. Vlasov, Phys. Lett. A, **111**, 401 (1985).
2. V. I. Kruglov, V. M. Volkov, R. A. Vlasov, and V. V. Drits, J. Phys. A. Math. Gen., **21**, 4381 (1988).
3. V. I. Kruglov, R. A. Vlasov, V. M. Volkov, and V. V. Drits, Vestsi Akad. Navuk BSSR, Ser. Fiz.-Mat. Navuk, No. 6, 105 (1989).
4. V. I. Kruglov, R. A. Vlasov, and V. M. Volkov, Izv. Akad. Nauk SSSR, Ser. Fiz., **53**, 1182 (1989).
5. V. I. Kruglov, "Interaction and Propagation of Electromagnetic Radiation in Crystalline and Gaseous Media: The Quantum Statistical Theory," Doctoral Dissertation, Minsk (1990).
6. V. I. Kruglov, Yu. A. Logvin, and V. M. Volkov, J. Mod. Opt., **39**, 2277 (1992).
7. V. M. Volkov, V. V. Drits, and V. I. Kruglov, Matematicheskoe Modelirovanie, **6**, 3 (19??).
8. E. M. Lifshits and L. P. Pitaevskii, Theoretical Physics [in Russian], Vol. 9, Moscow (1978).
9. R. P. Feynmann, Statistical Mechanics. A Set of Lectures, Los Angeles (1972).
10. D. Saint-Jamens, G. Sama, and E. J. Thomas, Type II Superconductivity, Oxford-London-Edinburgh-New York (1969).
11. V. I. Kruglov, in: Proc. of the IV International Workshop on Nonlinear Phenomena in Complex Systems, Minsk (1995).
12. V. I. Kruglov and V. M. Volkov, Orbitally Quantized Vortex States of the Electromagnetic Field of Laser Radiation, Preprint of the Physicotechnical Institute, Academy of Sciences of Belarus, Minsk (1995).
13. P. Coullet, L. Gil, and F. Rocca, Opt. Comm., 73, 403 (1989).
14. Z. Chen, N. B. Abraham, S. Balle, E. J. D'Angelo, and J. R. Tredicce, in: Chaos, Solitons, and Fractals, **4**, 1489-1517 (1994).
15. G. K. Harkness, J. C. Lega, and G. Oppo, in: Chaos, Solitons, and Fractals, **4**, 1519-1533 (1994).
16. V. M. Volkov and V. I. Kruglov, Differentsial'nye Uravneniya, **31**, 1184 (1995).
17. V. I. Kruglov and V. M. Volkov, in: Proc. of the IV International Workshop on Nonlinear Phenomena in Complex Systems, Minsk (1995).